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Pointwise Approximation by Baskakov Quasi-Interpolants

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Abstract—Recently, the quasi-interpolants of some classical operators were introduced. Mache and Müller gave Baskakov quasi-interpolants and obtained approximation equivalence theorem with $\omega_{\varphi}^{2r}(f,t)_{\infty}$. In this paper, we extend the above result with modulus $\omega_{\varphi^{\lambda}}^{2r}(f,t)_{\infty}$ $(0 \le \lambda \le 1)$ which unified classical modulus and Ditzian-Totik modulus.

Keywords—Baskakov quasi-interpolants, Pointwise approximation, Equivalence theorem, K-functional, Ditzian-Totik modulus.

1. INTRODUCTION

The Baskakov operator V_n, $n \in \mathbb{N}$ is given by

$$V_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad x \ge 0,$$

with $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$.

It is known that for $f \in C_B[0,\infty)$ (the set of bounded and continuous functions on $[0,\infty)$), $\varphi(x) = \sqrt{x(1+x)}$, $\alpha < 1$ (cf. [1, p.117 (9.3.3)])

$$\|V_n f - f\| = O\left(n^{-\alpha}\right) \Leftrightarrow \omega_{\varphi}^2(f,t) = O\left(t^{2\alpha}\right), \tag{1.1}$$

where $\omega_{\varphi}^2(f,t)$ is Ditzian-Totik modulus. In 1994, Ditzian used the unified modulus $\omega_{\varphi^{\lambda}}^2(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi^{\lambda}}^2 f\|$, $0 \le \lambda \le 1$ and obtained the direct estimate for Bernstein operator (cf. [2]), here $\varphi(x) = \sqrt{x(1-x)}$. The first article using φ^{λ} to bridge the gap between the classical moduli and the Ditzian-Totik moduli is Ditzian-Jiang [3]. Following this idea, we gave some approximation equivalence theorems for the linear combinations of Bernstein-type operators with $\omega_{\varphi^{\lambda}}^{2r}(f,t)$ (cf. [4,5]).

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Recently, the so-called left quasi-interpolants of some classical operators were investigated (cf. [6–10]). In [6], Mache and Mache obtained approximation equivalence theorem for Bernstein quasi-interpolants with $\omega_{\varphi}^{2r}(f,t)$. We extended their result to the case of $\omega_{\varphi^\lambda}^{2r}(f,t)$ in [7].

In this paper, we will consider the so-called left Baskakov quasi-interpolants $V_n(k,f,x)$, that is, for $0 \le k \le n$,

$$V_n(k,f,x) = \sum_{j=0}^{k} \alpha_j^n \left(D^j \circ V_n\right)(f,x) =: \sum_{j=0}^{k} \alpha_j^n V_{n,j}(f,x), \tag{1.2}$$

where $\alpha_j^n(x) \in \prod_j$ (the set of polynomial of order j), satisfying $V_n^{-1} = \sum_{j=0}^{n} \alpha_j^n D^j$, $D = \frac{d}{dx}$ on $\prod_n$.

In [8, p. 149, Thorem 4.4], the following equivalence result was proved.

THEOREM. *Let $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $n \ge 2r-1$, $r \in \mathbb{N}$, then for $0 < \alpha < r$, the following two statements are equivalent,*

$$\begin{aligned} &\text{(i)} && \|V_n(2r-1,f,x) - f(x)\|_\infty = O\left(n^{-\alpha}\right), \\ &\text{(ii)} && \omega_{\varphi}^{2r}(f,t)_\infty = O\left(t^{2\alpha}\right). \end{aligned} \tag{1.3}$$

In this paper, we will extend this result to the case of $\omega_{\varphi^\lambda}^{2r}(f,t)$ and get a pointwise approximation equivalence theorem as follows. For $0 \le \lambda \le 1$, $0 < \alpha < 2r$,

$$|V_n(2r-1,f,x) - f(x)| = O\left(\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\alpha}\right) \Leftrightarrow \omega_{\varphi^\lambda}^{2r}(f,t) = O(t^\alpha), \tag{1.4}$$

where $\varphi(x) = \sqrt{x(1+x)}$, $\delta_n(x) = \max\{\varphi(x), 1/\sqrt{n}\} \sim \varphi(x) + 1/\sqrt{n}$.

Obviously, (1.3) is the special case $\lambda = 1$ of (1.4).

Now, we give the definitions of the unified modulus and K-functional (cf. [1, p. 8,10 (2.1.1), p. 24 (3.1.1)])

$$\omega_{\varphi^\lambda}^{s}(f,t) = \sup_{0<h\le t} \sup_{x \ge (s/2)h\varphi^\lambda} \left|\Delta_{h\varphi^\lambda}^{s} f(x)\right|, \tag{1.5}$$

$$K_{\varphi^\lambda}^{s}(f,t^s) = \inf_{g \in W^s(\varphi,[0,\infty))} \left\{\|f-g\|_\infty + t^s \left\|\varphi^{s\lambda} g^{(s)}\right\|_\infty\right\}, \tag{1.6}$$

$$\bar{K}_{\varphi^\lambda}^{s}(f,t^s) = \inf_{g \in W^s(\varphi,[0,\infty))} \left\{\|f-g\|_\infty + t^s \left\|\varphi^{s\lambda} g^{(s)}\right\|_\infty + t^{s/(1-\lambda/2)} \left\|g^{(s)}\right\|_\infty\right\}, \tag{1.7}$$

where $W^s(\varphi,[0,\infty)) = \{g \mid g, g^{(s)} \in C_B[0,\infty), \|\varphi^{s\lambda} g^{(s)}\|_\infty < \infty\}$.

It was proved in [1, p. 11 (2.1.4) and p. 25 (3.1.4)] that

$$\omega_{\varphi^\lambda}^{s}(f,t) \sim K_{\varphi^\lambda}^{s}(f,t^s) \sim \bar{K}_{\varphi^\lambda}^{s}(f,t^s). \tag{1.8}$$

Throughout this paper, $\|\cdot\|$ denotes $\|\cdot\|_\infty$, C denotes a positive constant not necessarily the same at each occurrence.

2. DIRECT THEOREM

We will use the following results of [8].

LEMMA 2.1. *(See [8, p. 136, (2.2)–(2.8)].)*

(1) *For $j \ge 2$ and $x \in E_n^c = [0,1/n]$, there holds*

$$\left|\alpha_j^n(x)\right| \le Cn^{-j}, \tag{2.1}$$

$$\left|D^r \alpha_j^n(x)\right| \le Cn^{-j+r}. \tag{2.2}$$

(2) *For $j \ge 2$ and $x \in E_n = (1/n,\infty)$, there holds*

$$\left|\alpha_j^n(x)\right| \le Cn^{-j/2}\varphi^j(x), \tag{2.3}$$

$$\left|D^r \alpha_j^n(x)\right| \le Cn^{-j/2+r/2}\varphi^{j-r}(x). \tag{2.4}$$

THEOREM 2.2. *If* $\varphi(x) = \sqrt{x(1+x)}$, $\delta_n(x) = \max\{\varphi(x), 1/\sqrt{n}\}$, $0 \le \lambda \le 1$, $n \ge 2r-1$, *then for* $f \in C_B[0,\infty)$, *we have*

$$|V_n(2r-1,f,x) - f(x)| \le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.5}$$

REMARK 1. If $\lambda = 1$, then (2.5) is Theorem 3.1 in [8].

PROOF. By the definition of $\bar{K}_{\varphi^\lambda}^{2r}(f, t^{2r})$ for fixed n, x, λ, we can choose $g(t) = g_{\lambda,n,x}(t)$, such that

$$\begin{aligned} \|f-g\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r} \left\|\varphi^{2r\lambda} g^{(2r)}\right\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r/(1-\lambda/2)} \left\|g^{(2r)}\right\| \\ \le 2\bar{K}_{\varphi^\lambda}^{2r}\left(f, \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}\right). \end{aligned} \tag{2.6}$$

It is known that (cf. [8, p. 137, Lemma 2.3]) $\|V_n(k,f,x)\| \le M$, where $M > 0$ depends only on $k \in \mathbb{N}$.

As $V_n(k,f,x)$ is exact on $\prod_k$, i.e., $V_n(k,p,x) = p(x)$ for $p \in \prod_k$ (cf. [8, p. 133 line 21]), we have

$$\begin{aligned} |V_n(2r-1,f,x) - f(x)| &\le C\left(\|f-g\| + |V_n(2r-1,g,x) - g(x)|\right) \\ &= C\left(\|f-g\| + |V_n(2r-1, R_{2r}(g,\cdot,x),x)|\right) =: C\left(\|f-g\| + I\right), \end{aligned} \tag{2.7}$$

where $R_{2r}(g,t,x),x) = 1/((2r-1)!) \int_x^t (t-u)^{2r-1} g^{(2r)}(u)\,du$.

We only need to estimate I. As $\alpha_0^n(x) = 1$, $\alpha_1^n(x) = 0$ (cf. [8, p. 135, line 14]), we have

$$I \le |V_n(R_{2r}(g,\cdot,x),x)| + \left|\sum_{j=2}^{2r-1} \alpha_j^n(x) D^j V_n(R_{2r}(g,\cdot,x),x)\right| =: I_0 + \left|\sum_{j=2}^{2r-1} \alpha_j^n(x) I_j\right|. \tag{2.8}$$

To prove (2.5), we will show that

$$I_0 \le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{2.9}$$

and for $j = 2, \dots, 2r-1$,

$$\left|\alpha_j^n(x) I_j\right| \le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.10}$$

To estimate I_0 and I_j we consider two cases, Case 1 for $x \in E_n^c$ and Case 2 for $x \in E_n$.

CASE 1. $x \in E_n^c$. For $x \in E_n^c$, we have $\delta_n(x) \sim 1/\sqrt{n}$. It is easy to know (cf. [8, p. 140 (3.9)])

$$|R_{2r}(g,t,x)| \le \frac{1}{(2r-1)!} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| (t-x)^{2r} n^{r\lambda} \tag{2.11}$$

and (cf. [1, (9.5.10); 8, p. 137 (2.10)])

$$\left|V_n\left((t-x)^{2r}, x\right)\right| \le Cn^{-2r}, \tag{2.12}$$

we obtain

$$\begin{aligned} I_0 &\le Cn^{-2r} n^{r\lambda} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \\ &\le C\left[\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r} \left\|\varphi^{2r\lambda} g^{(2r)}\right\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r/(1-\lambda/2)} \left\|g^{(2r)}\right\|\right] \\ &\le C\omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \end{aligned} \tag{2.13}$$

For $x \in E_n^c$, using formula (cf. [1, (9.4.3)]),

$$V_{n,j}(f,x) = \frac{(n+j-1)!}{(n-1)!} \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^{j} f\left(\frac{k}{n}\right) p_{n+j,k}(x),$$

we have

$$\begin{aligned}
I_j &= \left|D^j V_n\left(R_{2r}(g,\cdot,x),x\right)\right| \\
&= \left|\frac{(n+j-1)!}{(n-1)!} \sum_{k=0}^{\infty} p_{n+j,k}(x) \overrightarrow{\Delta}_{1/n}^{j} \left(\frac{1}{(2r-1)!} \int_x^{k/n} \left(\frac{k}{n}-u\right)^{2r-1} g^{(2r)}(u)\,du\right)\right| \\
&\le \frac{(n+j-1)!}{(n-1)!} \sum_{k=0}^{\infty} p_{n+j,k}(x) \sum_{i=0}^{j} \binom{j}{i} \left|\int_x^{(k+i)/n} \left(\frac{k+i}{n}-u\right)^{2r-1} g^{(2r)}(u)\,du\right| \\
&\le Cn^j \sum_{k=0}^{\infty} p_{n+j,k}(x) \sum_{i=0}^{j} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \frac{((k+i)/n-x)^{2r}}{n^{-r\lambda}} \\
&\le Cn^j n^{r\lambda} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \sum_{i=0}^{j} \sum_{k=0}^{\infty} p_{n+j,k}(x) \left[\left(\frac{k}{n+j}-x\right)^{2r} + \left(\frac{k+i}{n}-\frac{k}{n+j}\right)^{2r}\right] \\
&\le Cn^j n^{r\lambda} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \sum_{k=0}^{\infty} p_{n+j,k}(x) \left[\left(\frac{k}{n+j}-x\right)^{2r} + \left(\frac{k}{n+j}\right)^{2r} + n^{-2r}\right] \\
&\le Cn^j n^{r\lambda} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \left(n^{-2r}+x^{2r}\right) \\
&\le Cn^j n^{r\lambda} n^{-2r} \left\|\delta_n^{2r\lambda} g^{(2r)}\right\| \le Cn^j \omega_{\varphi^\lambda}^{2r}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).
\end{aligned} \tag{2.14}$$

Here, we used that

$$\begin{aligned}
\sum_{k=0}^{\infty} p_{n,k}(x) \left(\frac{k}{n}\right)^{2r} &= \left(\sum_{k=0}^{2r} + \sum_{k=2r+1}^{\infty}\right) p_{n,k}(x) \left(\frac{k}{n}\right)^{2r} \\
&\le Cn^{-2r} + Cx^{2r} \sum_{k=2r+1}^{\infty} p_{n+2r,k-2r}(x) \le Cn^{-2r}.
\end{aligned}$$

With $|\alpha_j^n(x)| \le Cn^{-j}$ for $x \in E_n^c$ from (2.13) and (2.14), we have proved (2.9) and (2.10).

CASE 2. $x \in E_n$. For $x \in E_n$, $\delta_n(x) \sim \varphi(x)$, by the formula (cf. [1, p. 141]) for u between x and t,

$$|u-t|^{2r-1} u^{-r\lambda} (1+u)^{-r\lambda} \le |t-x|^{2r-1} x^{-r\lambda} (1+x)^{-(r-1)\lambda} \left(\frac{1}{(1+x)^\lambda} + \frac{1}{(1+t)^\lambda}\right),$$

then we have

$$|R_{2r}(g,t,x)| \le |t-x|^{2r} x^{-r\lambda} (1+x)^{-(r-1)\lambda} \left(\frac{1}{(1+x)^\lambda} + \frac{1}{(1+t)^\lambda}\right) \left\|\varphi^{2r\lambda} g^{(2r)}\right\|. \tag{2.15}$$

By [1, (9.4.14)] for $x \in E_n$, one has

$$V_n\left((t-x)^{2r}, x\right) \le Cn^{-r} \varphi^{2r}(x). \tag{2.16}$$

Using (2.15), we have

$$\begin{aligned} I_0 &= \left|V_n\left(R_{2r}\left(g,\cdot,x\right),x\right)\right| \\ &\le C\varphi^{-2r\lambda}\left(x\right)\left\|\varphi^{2r\lambda}g^{(2r)}\right\| V_n\left(\left(t-x\right)^{2r},x\right) \\ &\quad + C\varphi^{-(2r-2)\lambda}\left(x\right)x^{-\lambda}\left\|\varphi^{2r\lambda}g^{(2r)}\right\| V_n\left(\frac{\left(t-x\right)^{2r}}{\left(1+t\right)^{\lambda}},x\right). \end{aligned} \tag{2.17}$$

By [1, (9.6.3)] $V_n(1/(1+t)^m,x) \le C(m)1/(1+x)^m$, we get

$$\begin{aligned} V_n\left(\frac{\left(t-x\right)^{2r}}{\left(1+t\right)^{\lambda}},x\right) &\le \left(V_n\left(\left(t-x\right)^{4r},x\right)\right)^{1/2}\left(V_n\left(\frac{1}{\left(1+t\right)^{2}},x\right)\right)^{\lambda/2} \\ &\le Cn^{-r}\varphi^{2r}\left(x\right)\frac{1}{\left(1+x\right)^{\lambda}}. \end{aligned} \tag{2.18}$$

With (2.17) and (2.18), we get for $x \in E_n$

$$I_0 \le C\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}\left\|\varphi^{2r\lambda}g^{(2r)}\right\| \le C\omega_{\varphi^\lambda}^{2r}\left(f,\frac{\delta_n^{1-\lambda}\left(x\right)}{\sqrt{n}}\right). \tag{2.19}$$

For $x \in E_n$, using formula (cf. [1, p. 127] or [8, p. 138 (2.14)]),

$$\left|D^j p_{n,k}\left(x\right)\right| \le C\left\{\sum_{l=0}^{j}\left(\frac{\sqrt{n}}{\varphi\left(x\right)}\right)^{j+l}\left|\frac{k}{n}-x\right|^{l}\right\}P_{n,k}\left(x\right), \tag{2.20}$$

we have

$$\begin{aligned} |I_j| &= \left|D^j\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{1}{(2r-1)!}\int_x^{k/n}\left(\frac{k}{n}-u\right)^{2r-1}g^{(2r)}\left(u\right)du\right| \\ &\le C\sum_{k=0}^{\infty}\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi\left(x\right)}\right)^{j+i}\left|\frac{k}{n}-x\right|^{i}p_{n,k}\left(x\right)\left\|\varphi^{2r\lambda}g^{(2r)}\right\| \\ &\quad \cdot\frac{\left(k/n-x\right)^{2r}}{\varphi^{(2r-2)\lambda}\left(x\right)}\frac{1}{x^{\lambda}}\left(\frac{1}{\left(1+x\right)^{\lambda}}+\left(\frac{1}{\left(1+k/n\right)^{\lambda}}\right)\right) \\ &\le C\sum_{i=0}^{j}\left(\frac{\sqrt{n}}{\varphi\left(x\right)}\right)^{j+i}\left\|\varphi^{2r\lambda}g^{(2r)}\right\|\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\left|\frac{k}{n}-x\right|^{2r+i} \\ &\quad \cdot\left(\frac{1}{\varphi^{2r\lambda}\left(x\right)}+\frac{1}{\varphi^{(2r-2)\lambda}\left(x\right)x^{\lambda}\left(1+k/n\right)^{\lambda}}\right). \end{aligned}$$

As

$$\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\left|\frac{k}{n}-x\right|^{2r+i} \le Cn^{-r-i/2}\varphi^{2r+i}\left(x\right)$$

and

$$\begin{aligned} &\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\left|\frac{k}{n}-x\right|^{2r+i}\frac{1}{\left(1+k/n\right)^{\lambda}} \\ &\le\left(\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\left(\frac{k}{n}-x\right)^{4r+2i}\right)^{1/2}\left(\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{1}{\left(1+k/n\right)^{2}}\right)^{\lambda/2} \\ &\le Cn^{-r-i/2}\varphi^{2r+i}\left(x\right)\left(\frac{1}{1+x}\right)^{\lambda}. \end{aligned}$$

Hence, we have

$$|I_j| \le C\left(\frac{\sqrt{n}}{\varphi(x)}\right)^j \left\|\varphi^{2r\lambda} g^{(2r)}\right\| \left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r}.$$

With $|\alpha_j^n(x)| \le Cn^{-j/2}\varphi^j(x)$, we get

$$\left|\alpha_j^n(x) I_j\right| \le C\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^{2r} \left\|\varphi^{2r\lambda} g^{(2r)}\right\|.$$

By (2.13), (2.14), (2.19), and (2.21), we know that (2.9) and (2.10) hold, hence, from (2.6)–(2.8), we get (2.5).

3. INVERSE THEOREM

To prove inverse theorem, we need the following lemma.

LEMMA 3.1. *For $n \ge 2r-1$, $r \in \mathbb{N}$, $r \ge 2$, $0 \le \lambda \le 1$, then we have*

$$\left|\varphi^{2r\lambda}(x) D^{2r} V_n(2r-1, f, x)\right| \le Cn^r \delta_n^{2r(\lambda-1)}(x) \|f\| \qquad (f \in C_B[0,\infty)), \tag{3.1}$$

$$\left|\varphi^{2r\lambda}(x) D^{2r} V_n(2r-1, f, x)\right| \le C\left\|\varphi^{2r\lambda} f^{(2r)}\right\| \qquad \left(f \in W_\infty^{2r}(\varphi, [0,\infty))\right). \tag{3.2}$$

PROOF. Let us prove (3.1) first.

From [8, (4.1)] $\|\varphi^{2r} D^{2r} V_n(2r-1,f,x)\| \le Cn^r\|f\|$, for $x \in E_n$, we have

$$\begin{aligned}\left|\varphi^{2r\lambda}(x) D^{2r} V_n(2r-1, f, x)\right| &\le \varphi^{2r(\lambda-1)}(x) \left\|\varphi^{2r}(x) D^{2r} V_n(2r-1, f, x)\right\| \\ &\le Cn^r \varphi^{2r(\lambda-1)}(x) \|f\| \le Cn^r \delta_n^{2r(\lambda-1)}(x) \|f\|.\end{aligned}$$

For $x \in E_n^c$, from the procedure of the proof of [8, (4.1)], we have $|D^{2r}V_n(2r-1,f,x)| \le Cn^{2r}\|f\|$, noticing $\|\varphi^{2r\lambda}\|_{E_n^c} \sim n^{-r\lambda}$, we can easily get

$$\left|\varphi^{2r\lambda}(x) D^{2r} V_n(2r-1, f, x)\right| \le Cn^{2r} n^{-r\lambda}(x) \|f\| \le Cn^r \delta_n^{2r(\lambda-1)}(x) \|f\|.$$

So, we have (3.1).

Now, we prove (3.2). As $\alpha_0^n = 1$, $\alpha_1^n = 0$, and $\alpha_j^n \in \prod_j$ $(j \ge 2)$, we have for all $x \in [0, \infty)$,

$$\begin{aligned}\left|\varphi^{2r\lambda}(x) D^{2r} V_n(2r-1, f, x)\right| &= \varphi^{2r\lambda}(x) D^{2r}\left(\sum_{j=0}^{2r-1} \alpha_j^n(x) V_{n,j}(f,x)\right) \\ &= \varphi^{2r\lambda}(x) V_{n,2r}(f,x) \\ &\quad + \sum_{j=2}^{2r-1} \varphi^{2r\lambda}(x) \sum_{i=0}^{j} \binom{2r}{i} D^i \alpha_j^n(x) V_{n,2r+j-i}(f,x) \\ &=: \varphi^{2r\lambda}(x) V_{n,2r}(f,x) + S.\end{aligned} \tag{3.3}$$

We only estimate S in (3.3), for the other term $\varphi^{2r\lambda}(x)V_{n,2r}(f,x)$, the method is similar.

To estimate S, we write that for $x \in E_n^c$ (Case 1) (cf. [1, (9.4.3)]),

$$\begin{aligned}
|V_{n,2r+j-i}(f,x)| &= \left|D^{2r+j-i}V_n(f,x)\right| \\
&= \left|\frac{(n+2r+j-i-1)!}{(n-1)!}\sum_{k=0}^{\infty}\overrightarrow{\Delta}_{1/n}^{2r+j-i}f\left(\frac{k}{n}\right)p_{n+2r+j-i,k}(x)\right| \\
&\le Cn^{2r+j-i}\sum_{k=0}^{\infty}p_{n+2r+j-i,k}(x)\sum_{l=0}^{j-i}\binom{j-i}{l}\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k+l}{n}\right)\right| \\
&\le Cn^{2r+j-i}\left(\sum_{k=0}^{\infty}p_{n+2r+j-i,k}(x)\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|\right. \\
&\quad \left.+\sum_{k=0}^{\infty}p_{n+2r+j-i,k}(x)\sum_{l=1}^{j-i}\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k+l}{n}\right)\right|\right) \\
&=: Cn^{2r+j-i}(I_1+I_2).
\end{aligned} \tag{3.4}$$

Observing that (cf. [1, p. 155])

$$\begin{aligned}
\left|\left(\overrightarrow{\Delta}_{1/n}^{2r}f\right)\left(\frac{k}{n}\right)\right| &\le \begin{cases} Cn^{-2r+1}\displaystyle\int_0^{2r/n}\left|f^{(2r)}\left(\frac{k}{n}+u\right)\right|du, & \text{for } k\ge 1, \\ Cn^{-r+1}\displaystyle\int_0^{2r/n}u^r\left|f^{(2r)}(u)\right|du, & \text{for } k=0, \end{cases} \\
&\le \begin{cases} Cn^{-2r}\left\|\varphi^{2r\lambda}f^{(2r)}\right\|\left(\dfrac{k}{n}\right)^{-r\lambda}, & \text{for } k\ge 1, \\ Cn^{-r}\left\|\varphi^{2r\lambda}f^{(2r)}\right\|n^{-r(1-\lambda)}, & \text{for } k=0. \end{cases}
\end{aligned} \tag{3.5}$$

Therefore,

$$I_1 \le C\left(n^{-r}n^{-r(1-\lambda)}\left\|\varphi^{2r\lambda}f^{(2r)}\right\|+n^{-2r}\left\|\varphi^{2r\lambda}f^{(2r)}\right\|\sum_{k=1}^{\infty}p_{n+2r+j-i,k}(x)\left(\frac{k}{n}\right)^{-r\lambda}\right).$$

By a simple computation, it is easy to get for $\lambda \ne 0$,

$$\begin{aligned}
\sum_{k=1}^{\infty}p_{n+2r+j-i,k}(x)\left(\frac{n}{k}\right)^{r\lambda} &\le \left(\sum_{k=1}^{\infty}p_{n+2r+j-i,k}(x)\left(\frac{n}{k}\right)^{r}\right)^{\lambda} \\
&\le Cx^{-r\lambda}\le Cx^{-r\lambda}(1+x)^{-r\lambda}, \qquad \text{for } x\in E_n^c.
\end{aligned} \tag{3.6}$$

For $\lambda = 0$, (3.6) holds too.

Hence, we have

$$\begin{aligned}
I_1 &\le C\left\|\varphi^{2r\lambda}f^{(2r)}\right\|n^{-2r}\left(n^{r\lambda}+x^{-r\lambda}(1+x)^{-r\lambda}\right) \\
&\le Cn^{-2r}\delta_n^{-2r\lambda}(x)\left\|\varphi^{2r\lambda}f^{(2r)}\right\|\le Cn^{-2r}\varphi^{-2r\lambda}(x)\left\|\varphi^{2r\lambda}f^{(2r)}\right\|.
\end{aligned} \tag{3.7}$$

From above procedure for $k \ne 0$, similarly, we can deduce that

$$I_2 \le Cn^{-2r}\varphi^{-2r\lambda}(x)\left\|\varphi^{2r\lambda}f^{(2r)}\right\|. \tag{3.8}$$

Noticing that $|D^i\alpha_j^n(x)| \le Cn^{-j+i}$ for $x \in E_n^c$, by (3.3), (3.4), (3.7), and (3.8), we get

$$|S| \le C\varphi^{2r\lambda}(x)\,n^{-j+i}n^{2r+j-i}n^{-2r}\varphi^{-2r\lambda}(x)\left\|\varphi^{2r\lambda}f^{(2r)}\right\|\le C\left\|\varphi^{2r\lambda}f^{(2r)}\right\|, \qquad x\in E_n^c. \tag{3.9}$$

Next, we consider Case 2, $x \in E_n$.

If $0 < \lambda < 1$ we estimate S in (3.3). With (2.20) we have

$$\begin{aligned}
\varphi^{2r\lambda}(x)\left|V_{n,2r+j-i}(f,x)\right| &= \varphi^{2r\lambda}(x)\left|D^{j-i}V_{n,2r}(f,x)\right| \\
&= \varphi^{2r\lambda}(x)\left|D^{j-i}\frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}p_{n+2r,k}(x)\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right| \\
&\le C\varphi^{2r\lambda}(x)\cdot\frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{j-i}\left(\frac{\sqrt{n+2r}}{\varphi(x)}\right)^{j-i+l}\right. \\
&\quad \left.\cdot\left|\frac{k}{n+2r}-x\right|^{l}\right\}p_{n+2r,k}(x)\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right| \\
&\le C\varphi^{2r\lambda}(x)\cdot\frac{(n+2r-1)!}{(n-1)!}\sum_{l=0}^{j-i}\left(\frac{\sqrt{n+2r}}{\varphi(x)}\right)^{j-i+l} \\
&\quad \times\left\{\left(\sum_{k=0}^{\infty}\left|\frac{k}{n+2r}-x\right|^{l/1-\lambda}p_{n+2r,k}(x)\right)^{1-\lambda}\right. \\
&\quad \left.\times\left(\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|^{l/\lambda}p_{n+2r,k}(x)\right)^{\lambda}\right\} \qquad (3.10)\\
&= C\left(\frac{(n+2r-1)!}{(n-1)!}\right)^{1-\lambda}\sum_{l=0}^{j-i}\left(\frac{\sqrt{n+2r}}{\varphi(x)}\right)^{j-i+l} \\
&\quad \times\left\{\left(\sum_{k=0}^{\infty}\left|\frac{k}{n+2r}-x\right|^{l/1-\lambda}p_{n+2r,k}(x)\right)^{1-\lambda}\right. \\
&\quad \times\left(\sum_{k=0}^{\infty}\frac{(n+2r-1)!}{(n-1)!}\varphi^{2r}(x)\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|^{1/\lambda}\right. \\
&\quad \left.\left.\times p_{n+2r,k}(x)\right)^{\lambda}\right\} \\
&=: C\left(\frac{(n+2r-1)!}{(n-1)!}\right)^{1-\lambda}\sum_{l=0}^{j-i}\left(\frac{\sqrt{n+2r}}{\varphi(x)}\right)^{j-i+l}\{J_1\cdot J_2\}.
\end{aligned}$$

Noticing that (cf. [1, p. 153])

$$\begin{aligned}
&\frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\varphi^{2r}(x)p_{n+2r,k}(x)\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|^{1/\lambda} \\
&= n^{2r}\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|^{1/\lambda} \\
&\quad \times p_{n,k+r}(x)\left(\frac{k}{n}+\frac{1}{n}\right)\cdots\left(\frac{k}{n}+\frac{r}{n}\right)\left(1+\frac{k}{n}+\frac{r}{n}\right)\cdots\left(1+\frac{k}{n}+\frac{2r-1}{2}\right) \\
&\le C\left(n^{r}p_{n,r}(x)\left|\overrightarrow{\Delta}_{1/n}^{2r}f(0)\right|^{1/\lambda}+n^{2r}\sum_{k=1}^{\infty}p_{n,k+r}(x)\left(\frac{k}{n}\right)^{r}\left(1+\frac{k}{n}\right)^{r}\left|\overrightarrow{\Delta}_{1/n}^{2r}f\left(\frac{k}{n}\right)\right|^{1/\lambda}\right),
\end{aligned}$$

and $\varphi^{2r\lambda}(k/n) \le \varphi^{2r\lambda}(k/n+u)$ for $k>0$ and $u \ge 0$, we have with (3.5)

$$\begin{aligned} J_2 &\le C \left(n^r p_{n,r}(x) \left| \overrightarrow{\Delta}^{2r}_{1/n} f(0) \right|^{1/\lambda} \right. \\ &\quad \left. + n^{2r} \sum_{k=1}^{\infty} p_{n,k+r}(x) \left(\frac{k}{n}\right)^r \left(1+\frac{k}{n}\right)^r \left| \overrightarrow{\Delta}^{2r}_{1/n} f\left(\frac{k}{n}\right) \right|^{1/\lambda} \right)^{\lambda} \\ &\le C \left[n^r p_{n,r}(x) \left(n^{-r+1} \int_0^{2r/n} u^r f^{(2r)}(u)\, du \right)^{1/\lambda} \right. \\ &\quad \left. + n^{2r} \sum_{k=1}^{\infty} p_{n,k+r}(x) \left(n^{-2r+1} \int_0^{2r/n} \varphi^{2r\lambda}\left(\frac{k}{n}+u\right) \left| f^{(2r)}\left(\frac{k}{n}+u\right) \right| du \right)^{1/\lambda} \right]^{\lambda} \\ &\le C \left[n^r \left(n^{-r} n^{-r(1-\lambda)} \left\| \varphi^{2r\lambda} f^{(2r)} \right\| \right)^{1/\lambda} + n^{2r} \left(n^{-2r} \left\| \varphi^{2r\lambda} f^{(2r)} \right\| \right)^{1/\lambda} \right]^{\lambda} \\ &\le C n^{-2r(1-\lambda)} \left\| \varphi^{2r\lambda} f^{(2r)} \right\|. \end{aligned} \tag{3.11}$$

By [1, (9.4.14)] we can choose $q \in \mathbb{N}$, such that $2q(1-\lambda) > 1$, then we get

$$J_1 \le \left(\sum_{k=0}^{\infty} \left(\frac{k}{n+2r} - x \right)^{2ql} p_{n+2r,k}(x) \right)^{1/2q} \le C n^{-l/2} \varphi^l(x). \tag{3.12}$$

Together (3.10)–(3.12) for $x \in E_n$, we have

$$\left| \varphi^{2r\lambda}(x) V_{n,2r+j-i}(f,x) \right| \le C n^{2r(1-\lambda)} \left(\frac{\sqrt{n+2r}}{\varphi(x)} \right)^{j-i} n^{-2r(1-\lambda)} \left\| \varphi^{2r\lambda} f^{(2r)} \right\|. \tag{3.13}$$

Hence, using (2.4) $|D^j \alpha_j^n(x)| \le C n^{-j+i/2} \varphi^{j-i}(x)$, we obtain for $0 < \lambda < 1$,

$$|S| \le C \left\| \varphi^{2r\lambda} f^{(2r)} \right\|, \qquad x \in E_n. \tag{3.14}$$

From above precedure, we know that for the case of $\lambda = 0$ (the case of $\lambda = 1$ is similar), we need not use Hölder inequality in (3.10), it is easy to get (3.14).

By (3.9) and (3.14), we obtain that

$$|S| \le C \left\| \varphi^{2r\lambda} f^{(2r)} \right\|.$$

Similarly (but more simply), we can deduce

$$\left| \varphi^{2r\lambda}(x) V_{n,2r}(f,x) \right| \le C \left\| \varphi^{2r\lambda} f^{(2r)} \right\|. \qquad \blacksquare$$

THEOREM 3.2. *Let $f \in C_B[0,\infty)$, $n \ge 2r-1$, $r \in \mathbb{N}$, $0 \le \lambda \le 1$, $0 < \alpha < 2r$, then we have*

$$|V_n(2r-1,f,x) - f(x)| = O\left(\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right)^{\alpha} \right)$$

implies

$$\omega^{2r}_{\varphi^\lambda}(f,t) = O(t^\alpha).$$

PROOF. By Lemma 3.1, the proof of Theorem 3.2 is similar to [4, p. 145 " $\Leftarrow$ "], we omit the details. $\blacksquare$

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